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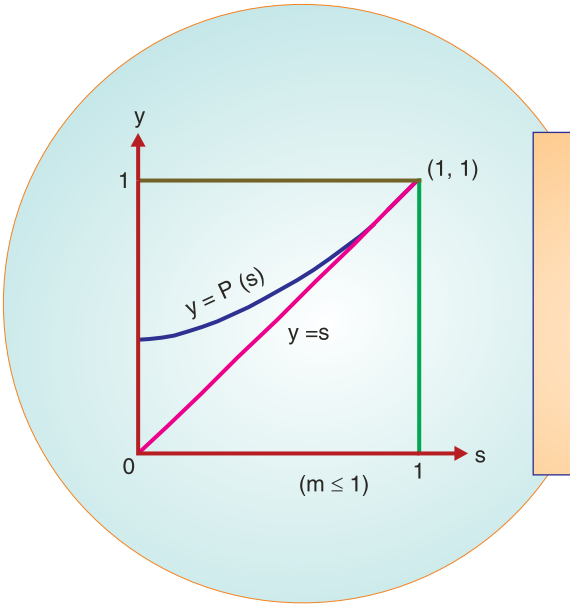

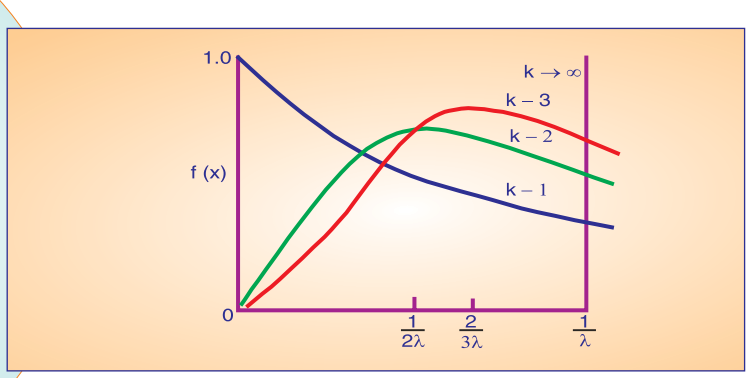

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SIXTH EDITION

J MEDHI
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Committed to Educate the Nation

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Preface to the Sixth Edition

I feel very honoured to find that the book first published in 1982 by Halsted Press, J. Wiley & Son, New York, USA and Wiley Eastern Ltd, New Delhi, India has been under print with new Editions (including International Editions) for four decades now.

The Review of the First Edition of 1982 that appeared in the reputed Journals

(a) *American Math. Monthly*, USA (Vol. 99#10, December, 1982, p. 99) &

(b) *Inst. of Math. Stat*. Bulletin (USA) (Vol. 66, January, 1983, p. 14).

noted as follows:

"Of the dozen or more texts published in the last five years aimed at students with a background of a first course in probability and statistics but not yet to measure theory, this is the clear choice."

"An extremely well organized, lucidly written text with numerous problems, examples and references": (TAV).

I would like to refer to Another Review of the 1982 First Edition: it was by Haward Taylor, of the reputed Cornell University, USA; it appeared in the US Journal *Technometrics* Vol. 26 No. 4, November, 1983: it commented as follows:

"The exposition is clear and competent." There are many typographical errors, but no major errors were noted by this Reviewer."

"In summary, this is a competent book treating the major traditional areas of Stochastic Processes at a very introductory to intermediate level, and deserves consideration by teachers of courses at this level."

There were other favourable Reviews for Second Edition as well. I would now like to refer to the Review of the Third Edition by Susan Pitts of the University of Cambridge in *J. American Statistical Association*, Vol. 106 (495), Sept 01, 2011 (pp. 1223-4).

Ms Shane White commented as follows:

"A good Review although highlighting the need for careful editing."

I am very much thankful to the *J. American Statistical Association*, Reviewer Professor Susan Pitts for drawing my attention to the portions that need more 'careful editing'.

She examined both the Second and Third Editions. In this Sixth Edition, I have tried to examine the portions that needed "careful editing" and to take appropriate action. I believe, with these, the Sixth Edition, would be free of those errors indicated by the Reviewer.

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I have also added some new material in the Appendix. I hope the new Sixth Edition would be more attractive to the graduate students, professionals and researchers.

H.C. Tijms and J.W. Hogenkamp (Vrieje University, Amsterdam) commented "He (J. Medhi) has made many important contributions to the field of applied probability and stimulated others to study this field by writing a nice book on Stochastic Processes" (*Probability Models* and *Statistics*: *J. Medhi Festschrift* (p. 69), New Age International Publishers, 1996).

I hope the Sixth Edition would be a step further towards this goal.

Guwahati, India **JYOTIPRASAD MEDHI**

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1

Random Variables and Stochastic Processes

1.1 GENERATING FUNCTIONS

1.1.1 Introduction

In dealing with integral-valued random variables, it is often of great convenience to apply the powerful tool of generating functions. Many stochastic processes that we come across involve non-negative integral-valued random variables, and quite often we could use generating function in their studies. The principal advantage of its use is that a single function (generating function) may be used to represent a whole set of individual items.

Definition. Let a_0, a_1, a_2, ... be a sequence of real numbers. Using a variable s, we may define a function

$$A(s)=a_0+a_1s+a_2s^2+\cdots=\sum_{k=0}^{\infty}a_ks^k. \tag{1.1}$$

If this power series converges in some interval $-s_0 < s < s_0$, then $A(s)$ is called the generating function (or the *s-transform*) of the sequence, a_0, a_1, a_2,.... The variable s itself has no particular significance. Here we assume s to be real but a generating function with a complex variable z is also sometimes used. It is then called a z-transform. The power series (1.1) can be differentiated any number of times, also $\frac{d}{ds}\sum=\sum\frac{d}{ds}$ holds, within its radius of convergence.

Differentiating (1.1) k times, putting $s = 0$ and dividing by $k!$, we get a_k, *i.e.*

$$a_k=\frac{1}{k!}\left[\frac{d^kA}{ds^k}\right]_{s=0} \tag{1.2}$$

1.1.2 Probability Generating Function: Mean and Variance

Suppose that X is a random variable that assumes non-negative integral values 0, 1, 2,..., and that

$$\Pr\{X = k\} = p_k, \quad k = 0, 1, 2, ..., \sum_k p_k = 1. \tag{1.3}$$

If we take a_k to be the probability p_k, $k = 0, 1, 2,...$ then the corresponding generating function $P(s) = \sum p_k S^k$ of the sequence of probabilities $\{p_k\}$ is known as the *probability generating function* (p.g.f.) of the random variable X. It is sometimes also called the s-transform (or *geometric transform*) of the r.v. X.

We have $P(1) = 1$; the series $P(s)$ converges absolutely and uniformly in the interval $-1 \leq s \leq 1$ and is infinitely differentiable. The function $P(s)$ is defined by $\{p_k\}$ and in turn defines $\{p_k\}$ uniquely, *i.e.* a p.g.f. determines a distribution uniquely. Again

$$P(s) = \sum_{k=0}^{\infty} \Pr\{X = k\} s^k = E\left(s^X\right), \tag{1.4}$$

where $E(s^X)$ is the expectation of the function s^X (a random variable) of the random variable X.

The first two derivatives of $P(s)$ are given by

$$P'(s) = \sum_{k=1}^{\infty} k p_k s^{k-1}, \; -1 < s < 1 \tag{1.5a}$$

and

$$P''(s) = \sum_{k=1}^{\infty} k(k-1) p_k s^{k-2}. \tag{1.5b}$$

The expectation $E(X)$ is given by

$$E(X) = \sum_{k=1}^{\infty} k p_k = \lim_{s \to 1} P'(s) = P'(1). \tag{1.6}$$

We have

$$E\{X(X-1)\} = \sum_k k(k-1) p_k = \lim_{s \to 1} P''(s) \tag{1.7}$$

and

$$E\left(X^2\right) = E\{X(X-1)\} + E(X) = P''(1) + P'(1).$$

Hence

$$\text{var}(X) = E\left(X^2\right) - \left[E(X)\right]^2$$

$$= P''(1) + P'(1) - \left[P'(1)\right]^2 \tag{1.8}$$

If, as $s \to 1$, $\sum k p_k$ diverges then we say that $E(X) = P'(1) = \infty$ and if $\sum k(k-1) p_k$ diverges then $P''(1) = \infty$ and var $(X) = \infty$.

The relation (1.6) gives the mean and (1.8) the variance of X in terms of the p.g.f. of X. In fact, moments, cumulants etc. can be expressed in terms of generating functions. More generally, the kth factorial moment of X is given by

$$E\{X(X-1)...(X-k+1)\} = \frac{d^k}{ds^k} P(s)\Big|_{s=1} \text{ for } k = 1, 2,...$$

Note that $P(e^t)$ is the moment generating function and $P(1 + t)$ is the factorial moment generating function. Further, the p.g.f. $P(s) = E(s^X)$ is a special case of the characteristic function $E(e^{itX})$ or the Fourier transform of X.

Note 1. Assuming that term by term integration is valid, we get

$$\int_0^1 P(s)ds = E\left(\frac{1}{X+1}\right).$$

See also Exercise 1.6.

Note 2. Let $q_k = \Pr(X > k) = p_{k+1} + p_{k+2} + \ldots$, $k = 0, 1, 2\ldots$

and
$$r_k = \frac{\Pr(X > k-1)}{k}, \; k = 1, 2,\ldots$$

Then

$$Q(s) = \sum_{k=0}^{\infty} q_k s^k = \frac{1-P(s)}{1-s}, \qquad s \neq 1$$

and
$$R(s) = \sum_{k=1}^{\infty} r_k s^k = \int_0^s Q(y)\, dy.$$

Example 1(a). *Bernoulli Distribution:* Let X be a r.v. that assumes only two values 0 and 1 with

$$P\{X = 1\} = p,\; P\{X = 0\} = q = 1 - p\,(0 < p < 1).$$

The p.g.f. of the r.v. X is

$$P(s) = E\left[s^X\right] = q + sp.$$

We have $E(X) = p$ and var $(X) = pq$. Independent trials with only two possible outcomes ("success" and "failure") at each trial are known as Bernoulli trials. Denote "success" by 1 and "failure" by 0 and their probabilities by p and q respectively; we then get Bernoulli distribution and X is called a Bernoulli r.v. with parameter p.

Example 1(b): *Binomial Distribution:* Let X be a r.v. denoting the number of successes (or failures) in a fixed number n of Bernoulli trials. Then X has binomial distribution having p.m.f.

$$p_k = \Pr(X = k) = \binom{n}{k} p^k q^{n-k}, \qquad k = 0, 1, 2,\ldots, n$$

The p.g.f. of X is

$$P(s) = (q + sp)^n$$

with $E(X) = np$ and var $(X) = npq$.

The binomial r.v. has parameters n and p and is the sum of n independent Bernoulli r.v.'s each with parameter p.

Example 1(c). *Poisson Distribution:* Let X be a Poisson variate with p.m.f.

$$p_k = \Pr\{X = k\} = \frac{\exp(-\lambda)\lambda^k}{k!},\ k = 0, 1, 2,...$$

The p.g.f. of Poisson distribution (or of the random variable X) is given by

$$P(s) = \sum_{k=0}^{\infty} p_k s^k = \sum_{k=0}^{\infty} \frac{\exp(-\lambda)\lambda^k}{k!} s^k$$

$$= \exp(-\lambda)\exp(\lambda s)$$

$$= \exp\{\lambda(s-1)\}.$$

Since $P'(s) = \lambda \exp\{\lambda(s-1)\}$ and $P''(s) = \lambda^2 \exp\{\lambda(s-1)\}$, we have, from (1.6) and (1.8)

$$E(X) = \lambda \text{ and } \operatorname{var}(X) = \lambda.$$

Note that $p_{k+1} = \frac{\lambda}{k+1} p_k,\ k \geq 0;$ this recursive formula can be used to compute Poisson probabilities starting with $p_1 = \lambda p_0 = \lambda e^{-\lambda}$.

Example 1(d). *Geometric Distribution:* Let X be a random variable with geometric distribution $\{p_k\}$, where

$$p_k = \Pr\{X = k\} = q^k p, \qquad k = 0, 1, 2, ...$$

$$0 < q = 1 - p < 1.$$

The p.g.f. of X is

$$P(s) = \sum_{k=0}^{\infty} p_k s^k = p \sum_{k=0}^{\infty} (qs)^k = \frac{p}{1-qs}.$$

We have

$$P'(s) = \frac{pq}{(1-qs)^2},\ \ P''(s) = \frac{2pq^2}{(1-qs)^3}.$$

Using (1.6) and (1.8) we have

$$E(X) = P'(1) = \frac{pq}{(1-q)^2} = \frac{q}{p}$$

and

$$\operatorname{var}(X) = P''(1) + P'(1) - [P'(1)]^2 = \frac{2q^2}{p^2} + \frac{q}{p} - \left(\frac{q}{p}\right)^2 = \frac{q}{p^2}.$$

An example of a random variable X that follows geometric distribution is given below. Consider a sequence of Bernoulli trials with probability of success p and that of failure $q = 1 - p$ at each trial. Then

the number X of failures preceding the first success has geometric distribution with $\Pr\{X = k\} = q^k p$ (k = 0, 1, 2,...). The r.v. X is also called the waiting time for the first success.

Non-ageing property of geometric distribution

For a geometric r.v. X, we have

$$\Pr\{X = s + r \mid X \geq s\} = \frac{q^{s+r} p}{q^s}$$

$$= q^r p = \Pr\{X = r\}.$$

This property, called non-ageing (or memoryless) property, characterizes geometric distribution among all distributions of discrete non-negative integral r.v.'s.

Note: If $Y_1, Y_2,...$, is a sequence of independent Bernoulli *r.v.s*, then

$X_i = \min\{i, Y_i = 1\} - 1$ is a geometric r.v.

Example 1(e). *Logarithmic Series Distribution:*

The r.v. X has logarithmic series distribution if

$$p_k = \Pr\{X = k\} = \frac{\alpha q^k}{k}, \quad k = 1, 2, 3, ...$$

$$\alpha = -1/(\log p)$$

$$0 < q = 1 - p < 1.$$

The p.g.f. of X is

$$P(s) = \sum_{k=1}^{\infty} p_k s^k = \sum_{k=1}^{\infty} \frac{\alpha q^k}{k} s^k$$

$$= -\alpha \log(1 - sq) = \frac{\log(1 - sq)}{\log(1 - q)}.$$

We have

$$E(X) = \frac{\alpha q}{1 - q}$$

and

$$\text{var}(X) = \frac{\alpha q(1 - \alpha q)}{(1 - \alpha)^2}.$$

Example 1(f). Let X be a random variable with p.g.f. $P(s)$. To find the p.g.f. of the random variable $Y = mX + n$, where m, n are integers and $m \neq 0$.

Let

$$P_X(s) \text{ and } P_Y(s)$$

be the generating functions of X and Y respectively. We have

$$P_Y(s) = E\{s^Y\} = E\{S^{mX+n}\}$$

$$= E\{s^{mX} \cdot s^n\} = s^n \cdot E\{(s^m)^X\}$$

$$= s^n P_X(s^m).$$

Example 1(g). *Indicator Random Variable.* Let A be an event; then the indicator of A is the random variable

$$I(A) = \begin{cases} 1 & \text{if the event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

It is a Bernoulli random variable having parameter $p = \Pr\{A\} = E\{I(A)\}$.

1.1.2.1 Determination of $\{p_k\}$ from a given $P(s)$

From the above examples, we see how a single generating function $P(s)$ may be used to represent a whole set of probabilities

$$p_k = \Pr\{X = k\}, \quad k = 0, 1, 2, ...$$

In these examples we were concerned with the problem of finding $P(s)$ for a given set of p_k's. In many cases the reverse problem arises: to determine p_k from a given p.g.f. $P(s)$. Many situations arise, where it is easier to find the p.g.f. $P(s)$ of a variable rather than the probability distribution $\{p_k\}$ of the variable. One proceeds to find first the p.g.f. $P(s)$ and then to find the probability p_k from the function $P(s)$. Even without finding the p_k's one can find the moments of the distribution from $P(s)$.

Again, p_k can be (uniquely) determined from $P(s)$ as follows:

p_k can be found from $P(s)$ by applying (1.2), *i.e.*

$$p_k = \frac{1}{k!}\left[\frac{d^k P(s)}{ds^k}\right]_{s=0};$$

p_k is also given by the coefficient of s^k in the expansion of $P(s)$ as a power series in s.

When $P(s)$ is of the form $P(s) = U(s)/V(s)$, it may be convenient to expand $P(s)$ in a power series in s first by decomposing $P(s)$ into partial factions. Suppose that $s_1, ..., s_r$ are the *distinct roots* of $V(s)$, *i.e.* $V(s) = (s - s_1)... (s - s_r)$ apart from a constant factor c, which, for simplicity, we take to be equal to 1), then $P(s)$ can be decomposed into partial fractions as

$$P(s) = \frac{a_1}{s_1 - s} + \cdots + \frac{a_r}{s_r - s},$$

where a_i's can be determined. It may be verified that

$$a_i = -U(s_i)/V'(s_i).$$

Now

$$\frac{a_i}{s_i - s} = \frac{a_i}{s_i(1 - s/s_i)} = \frac{a_i}{s_i}\left(1 - \frac{s}{s_i}\right)^{-1}$$

$$= \frac{a_i}{s_i}\left\{1 + \frac{s}{s_i} + \left(\frac{s}{s_i}\right)^2 + \circ\right\} \text{ for } |s| < |s_i|$$

$$= a_i \sum_{k=0}^{\infty} \frac{s^k}{s_i^{k+1}}.$$

Hence $p_n \equiv$ coefficient of s^n in $P(s)$

$$= \frac{a_1}{s_1^{n+1}} + \circ \; + \frac{a_i}{s_i^{n+1}} + \circ \; + \frac{a_r}{s_r^{n+1}}.$$

The above is an exact expression for p_n. The calculation of p_n by the above expression involves determination of all the roots of $V(s)$. A simple approximation can be had as follows.

Suppose that s_1 is a root which is smaller in absolute value than all the other roots. Then as n increases, the first term becomes the dominant term, and the contributions of the other terms become negligible. Thus, as $n \to \infty$

$$p_n \cong \frac{a_1}{s_1^{n+1}}$$

The case when $V(s)$ has *multiple roots* and be treated in a similar manner. Suppose that s_1 is a root of order m; then we shall have

$$a_1 = \lim_{s \to s_1} \left[(s - s_1)^m P(s)\right]$$

$$= \frac{(-1)^m m! U(s_1)}{V^m(s_1)}.$$

If s_1 is of smaller absolute value than the other roots of $V(s)$, we shall get as approximation, as $n \to \infty$,

$$P_n \cong \frac{a_1}{s_1^{n+m}} \binom{n+m-1}{m-1}.$$

Note: Numerical inversion of PGF. Techniques of numerical inversion of PGFs are often taken recourse to in practice when the analytical method is not found suitable. For this, discrete Fast Fourier Transform (FFT) method is a very powerful method. This method has many other applications as well.

1.1.3 Sum of (a Fixed Number of) Random Variables

Let X and Y be two independent non-negative integral-valued random variables with probability distributions given by

$$\Pr\{x = k\} = a_k, \Pr\{Y = j\} = b_j.$$

The sum $Z = X + Y$ is a random variable. The event $\{Z = r\}$ can happen in the following mutually exclusive ways with corresponding probabilities:

$(X = 0 \text{ and } Y = r)$ with probability $a_0 b_r$

$(X = 1 \text{ and } Y = r-1)$ with probability $a_1 b_{r-1}$

... ...

$(X = t \text{ and } Y = r-t)$ with probability $a_i b_{r-t}$

... ...

$(X = r \text{ and } Y = 0)$ with probability $a_r b_0$.

Hence the distribution of Z is given by

$$c_r = \Pr\{Z=r\} = a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0 = \sum_{t=0}^{r} a_t b_{r-t}. \tag{1.9}$$

This new sequence $\{c_r\}$ that results from a combination of the two sequences $\{a_k\}$, $\{b_j\}$ in this particular way indicated by (1.9) is called the *convolution* of $\{a_k\}$ and $\{b_j\}$ and is denoted by

$$\{c_r\} = \{a_r\} * \{b_r\}.$$

Let $A(s)$, $B(s)$, $C(s)$ be the p.g.f. of X, Y and Z respectively.

Then from (1.9) it follows that

$$C(s) = A(s) \cdot B(s) \tag{1.9a}$$

This also follows from the fact that

$$C(s) = E\left(s^{X+Y}\right) = E\left(s^X\right) E\left(s^Y\right),$$

$$= A(s) \cdot B(s),$$

because of independence of X, Y.

We have thus the following theorem:

Theorem 1.1. The p.g.f. of the sum of two *independent* random variables X and Y is the product of the p.g.f. of X and that of Y.

The result will also hold good in the case of the sum S_n of n non-negative independent integral-valued random variables $X_1,..., X_n$, *i.e.* the p.g.f. of S_n is the product of the p.g.f's of $X_1, X_2,...,X_n$.

In particular, we have the following:

Theorem 1.2. The sum $S_n = X_1 + \ + X_n$ of a *fixed* number n of *identically and independently distributed* (i.i.d.) random variables X_i has the p.g.f. $\{P(s)\}^n$, each X_i having p.g.f. $P(s)$.

The results stated above are of considerable importance. These can be employed to find the distribution of the sum of a number of independent random variables.

It can be easily verified that if X_i are i.i.d. Bernoulli r.v.'s, then S_n has Binomial distribution.

Example 1(h). *Sum of Independent Poisson Variates:* Let X_1, X_2 be two independent Poisson variates with parameters (means) λ_1, λ_2 respectively. To find the distribution of the sum $Z = X_1 + X_2$.

The p.g.f. of X_i is $\exp\{\lambda_i (s-1)\}$, $i = 1, 2$.

Using (1.9a) the p.g.f. of Z is obtained as

$$\left[\exp\{\lambda_1(s-1)\}\right]\left[\exp\{\lambda_2(s-1)\}\right]$$

$$=\exp\{(\lambda_1+\lambda_2)(s-1)\}.$$

But this is the p.g.f. of a Poisson variate with parameter $\lambda_1 + \lambda_2$. Hence the sum of two independent Poisson variates is a Poisson variate. More generally, if X_i, $i = 1, 2, ..., n$ are n independent Poisson variates with parameters λ_i, then the sum $S = X_1 + X_2 + \cdots + X_n$ is a Poisson variate with parameter $\sum_{i=1}^{n}\lambda_i$

Example 1 (i). *Negative Binomial Distribution.* Let X_1, X_2 be two independent random variables each with the same geometric distribution, $\Pr(X_i = k) = q^k p$, $k = 0, 1, 2,...$

The p.g.f. of X_i is $p/(1 - qs)$ for $i = 1, 2$. Thus, the sum $Z = X_1 + X_2$ has the p.g.f.

$$P(s)=\left(\frac{p}{1-qs}\right)^2=p^2\sum_{r=0}^{\infty}\binom{r+1}{r}q^r s^r.$$

Hence

$$\Pr\{Z=r\}\equiv\text{coefficient of } s^r \text{ in } P(s)$$

$$=\binom{r+1}{r}q^r p^2=(r+1)q^r p^2,\qquad r=0, 1, 2, ...$$

This gives the distribution of the sum Z. If X_1 denotes the number of failures preceding the first success and X_2 the number of failures following the first success and preceding the second success, then $Z = X_1 + X_2$ denotes the number of failures preceding the second success.

The sum $S_r = X_1 + \ + X_r$ of r i.i.d. geometric variates denotes the number of failures preceding the rth success. It has the p.g.f. $P(s)=\left(\frac{p}{1-sq}\right)^r$. It follows that

$$\Pr\{S_r=k\}\equiv\text{coefficient of } s^k \text{ in } P(s)$$

$$=\binom{r+k-1}{k}p^r q^k,\qquad k=0, 1, 2, ... \tag{1.10}$$

and

$$\sum_k\binom{r+k-1}{k}p^r q^k=1. \tag{1.10a}$$

Here r is a positive integer. Now the quantity given by the r.h.s. of (1.10) is non-negative and (1.10a) holds for *any positive r*. The distribution (1.10) for $r > 0$ is known as *negative binomial distribution* with *index r,* and mean *rq/p*. In particular, when r is a positive integer, it is known as *Pascal distribution.*

(See Exercise 1.9 for distribution of the sum of two independent geometric variables).

Example 1(j). If X, Y are independent r.v.'s. with generating functions $A(s)$, $B(s)$ respectively, then the generating function $D(s)$ of $W = X - Y$ is given by

$$D(s) \equiv E\{s^{X-Y}\} = E\{s^X (s^{-1})^Y\} = E\{s^X\} E\{(1/s)^Y\},$$

because of independence of X, Y; thus Pr $\{X - Y = k\}$ is the coefficient of s^k, $k = 0. \pm 1, \pm 2, \ldots$ in the expansion of

$$D(s) = A(s) B(1/s).$$

1.1.4 Sum of a Random Number of Discrete Random Variables (Stochastic Sum)

In Section **1.1.3** we considered the sum $S_n = X_1 + \ + X_n$ of a fixed number of mutually independent random variables. Sometimes we come across situations when we have to consider the sum of *a random number N* of random variables. For example, if X_i denotes the number of persons involved in the ith accident (in a day in a certain city) and if the number N of accidents happening on a day is a random variable, then the sum $S_N = X_1 + \ldots + X_N$ denotes the total number of persons involved in accidents on a day.

We have the following theorem for the sum S_N.

Theorem 1.3. Let X_i, $i = 1, 2, \ldots$, be identically and independently distributed random variables with Pr $\{X_i = k\} = p_k$ and p.g.f.

$$P(s) = \sum_k p_k s^k \text{ for } i = 1, 2, \ldots \tag{1.11}$$

Let

$$S_N = X_1 + \cdots + X_N, \tag{1.12}$$

where N is a random variable **independent** of the X_i's. Let the distribution of N be given by Pr $\{N = n\} = g_n$ and the p.g.f. of N be

$$G(s) = \sum g_n s^n. \tag{1.13}$$

Then the p.g.f. $H(s)$ of S_N is given by the compound function $G(P(s))$, *i.e.*

$$H(s) = \sum_j \Pr\{S_N = j\} s^j = G(P(s)). \tag{1.14}$$

Proof: Since N is a random variable that can assume values 0, 1, 2, 3,..., the event $S_N = j$ can happen in the following mutually exclusive ways: $N = n$ and $S_n = X_1 + \cdots + X_n = j$ for $n = 1, 2, 3, \ldots$ To meet the situation $N = 0$ let us define $X_0 = S_0 = 0$ so that

$$X_0 + X_1 + \circ \ + X_n = S_N, \qquad n \geq 0.$$

We have $h_j = \Pr\{S_N = j\} = \sum_{n=0}^{\infty} \Pr\{N = n \text{ and } S_n = j\}.$

Since N is independent of X_i's and therefore of S_n,

$$h_j = \sum_{n=0}^{\infty} \Pr\{N = n\} \Pr\{S_n = j\} \tag{1.15}$$

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$$= \sum_{n=0}^{\infty} g_n \Pr\{S_n = j\}. \tag{1.16}$$

(The inclusion of the value 0 for n can be justified since Pr $\{S_0 = 0\} = 1$ and Pr $\{S_0 > 0\} = 0$).

The p.g.f. of the sum $S_N = X_1 + ... + X_N$ is thus given by

$$H(s) = \sum_j h_j s^j = \sum_j \Pr\{S_N = j\} s^j \tag{1.17}$$

$$= \sum_{j=0}^{\infty} \left[\left\{ \sum_{n=0}^{\infty} g_n \Pr\{S_n = j\} \right\} \right] s^j \tag{1.18}$$

$$= \sum_n \left[\sum_j \Pr\{S_n = j\} s^j \right] g_n$$

$$= \sum_n g_n \left[P(s) \right]^n = G(P(s)),$$

since the expression $\{P(s)\}^n$ is the p.g.f. of the sum S_n of a fixed number n of i.i.d. random variables (Theorem 1.2).

As $P(s)$ and $G(s)$ are p.g.f.'s, so also is $G(P(s))$; $G(P(s)) = 1$ for $s = 1$.

Since $G(P(s))$ is a compound function, the corresponding distribution is known as *compound distribution.* It is also known as *random sum distribution.*

Mean and Variance of S_N

We have

$$E(S_N) = H'(s)\big|_{s=1} = \left[P'(s) G'(P(s)) \right]_{s=1}$$

$$= P'(1) G'(1) (P(1)) = P'(1) G'(1)$$

so that

$$E\{S_N\} = E\{X_i\} E\{N\}. \tag{1.19}$$

Again,

$$H''(s) = P''(s) G'(P(s)) + G''(P(s)) [P'(s)]^2$$

Thus

$$E(S_N^2) - E(S_N) = H''(s)\big|_{s=1}$$

$$= \left[E(X_i^2) - E(X_i) \right] E(N) + \left[E(N^2) - E(N) \right] \left[E(X_i) \right]^2$$

Using (1.19) we get

$$E(S_N^2) = E(X_i^2) E(N) - E(N) \left[E(X_i) \right]^2 + E(N^2) \left[E(X_i) \right]^2$$

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$$= E(N)\text{var}(X_i) + E(N^2)\left[E(X_i)\right]^2$$

Finally,

$$\text{var}(S_N) = E(S_N^2) - \left[E(S_N)\right]^2$$

$$= E(N)\text{var}(X_i) + \left[E(X_i)^2\right]\text{var}(N). \qquad (1.20)$$

Note 1: See also § 1.15 for an alternative proof.

Note 2: Denote the k^{th} central moment of Y as

$$\mu_k(Y) = E\left[Y - E(Y)\right]^k, \quad k = 1, 2, 3, \ldots$$

Then the 3rd central moment of S_N is given by

$$\mu_3(S_N) = E(N)\mu_3(X_i) + 3E(X_i)\mu_2(X_i)\mu_2(N) + \left[E(X_i)\right]^3 \mu_3(N).$$

Example 1(k). The number of patients that visit a doctor's clinic (without appointment) is a Poisson r.v. with mean λ. The time taken by the doctor on a patient, is independently and uniformly distributed over $(0, h)$ [hours].

To find the mean and variance of the time taken by the doctor to complete consultation of all the patients.

Here N is Poisson with mean and variance λ. The time taken X_i by a patient (independent of the number of patients) is *i.i.d.* uniform over $(0, h)$ [hours]. Thus the total time taken by the patients is

$$S_N = \sum_{i=1}^{N} X_i.$$

We have

$$E(S_N) = E(N)E(X_i) = \lambda \cdot \frac{h}{2}$$

and

$$\text{var}(S_N) = \lambda \cdot \frac{h^2}{12} + \left(\frac{h}{2}\right)^2 \cdot \lambda = \frac{\lambda h^2}{3}.$$

For $\lambda = 12, h = \frac{1}{2}$ hour, mean is 3 hours and s.d. 1 hour.

Example 1(l). *Compound Poisson Distribution:* If N has a Poisson distribution with mean λ, then $G(s) = \sum g_n S^n = \exp\{\lambda(s - 1)\}$ and hence the sum $X_1 + \cdots + X_N$ has p.g.f.

$$H(s) = G(P(s)) = \sum g_n \left[P(s)\right]^n = \exp\left\{\lambda\left[P(s) - 1\right]\right\}.$$

The distribution having a generating function of the form $\exp\{\lambda[P(s) - 1]\}$, where $P(s)$ is itself a generating function, is called *Compound Poisson Distribution.*

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Jyotiprasad Medhi, Former Professor Emeritus, Gauhati University, Guwahati, India, was a Visiting Professor at the Universities of Montreal, McMaster and Toronto (Canada), and University of Wisconsin (USA). He has been invited to deliver lectures at several universities in India and abroad. In addition to publishing a large number of research papers and an introductory text (for Social Sciences etc.) *Statistical Methods*: *An Introductory Text* 2nd edition (2006), he has authored two **advanced level** books: *Stochastic Processes* 6th edition (2023) and *Introduction to Queueing Systems and Applications* (2014).

Glimpses into the Life and Works of J Medhi (authored by C K Chetia and published by Vigyan Prasar, an autonomous organisation under the Department of Science & Technology, Government of India, New Delhi), gives an account of his contributions.

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